THURSDAY, DECEMBER 27, 1906.

THE THEORY OF AGGREGATES.

The Theory of Sets of Points. By W. H. Young and G. C. Young. Pp. xii+316. (Cambridge: University Press, 1906.) Price 12*s.* net.

CONDESCENDING to a pun, Gauss once remarked that he was more interested in notions than in notations. The theory of aggregates is so independent of the ordinary symbolism of mathematics that it requires hardly any previous acquaintance with other branches of the science from those who proceed to the study of it. At the same time, it is full of peculiar difficulties: it abounds in seeming paradoxes; and some of its fundamental problems are at the present time the subject of keen research and controversy. A hearty welcome is therefore due to a work composed by authors who are familiar with all that has been published about aggregates, and have themselves made important contributions to the subject.

It is impossible to go into detailed criticism of this treatise without the use of technical terms which would convey no meaning to the ordinary reader; but an attempt may be made to show the general nature of this novel theory and the influence it has had, and will extend, over the first principles of other parts of mathematics.

The names of Cantor and Dedekind will always be associated with the first truly logical definition of the arithmetical continuum, or, which comes to the same thing, of the range of a real arithmetical variable. It is hardly possible to lay too much emphasis on the fact that all strictly arithmetical operations are connected with the elements (rational and irrational) of this continuum. Cantor's transcendental numbers obey laws of operation different from those of ordinary arithmetic, and the calculus associatfd with them ought to have another name.

With the help of a postulate, the necessity of which was first realised by Cantor and Dedekind, we can establish a one-one correspondence between the values of a real positive variable and the points of a finite straight line, exclusive of one end, if the variable is not allowed to be infinite. If we extend the postulate so as to include both ends of the segment, we have to include values 0 and ∞ for the variable, which, from this point of view, are equally definite. For convenience, we speak of a point x, instead of saying "a point corresponding to the number x."

From the arithmetical side we have to investigate the properties of the continuum and of its parts; and the special interest of the subject begins when we consider parts which contain more than a finite set of elements. The simplest of these is the natural scale 1, 2, 3, &c.; its characteristic properties are that it has a natural order with a first element, each element being succeeded by the next higher in magnitude, and there being no last element.

The set of rational numbers differs from the natural scale in some very important respects. As represented by points on a line, they have an order of position, corresponding to their order of size; but we cannot say that in this order any element is followed by a "next" element. In fact, between any two distinct rational points lie an infinite set of other rational points. But Cantor was the first to point out that the rational set may be brought into a one-one correspondence with the natural scale, for example, in the order:—

$$, \tfrac{2}{1}, \tfrac{1}{2}, \tfrac{3}{1}, \tfrac{1}{3}, \tfrac{4}{1}, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{1}{4}, \text{\&c.},$$

where the fractions, in their lowest terms, are arranged so that the sum of numerator and denominator never diminishes, while those with the same sum are placed in descending order of magnitude. Every set which can be thus brought into correspondence with the natural scale is said to be countably infinite, or to be of potency a. Examples of such sets are: (1) the set of all algebraic numbers, (2) all points with rational coordinates in a space of n dimensions, where n is any assigned positive integer.

On the other hand, the arithmetical continuum is not countably infinite, and its potency, denoted by c, is of a higher order than a. One of the outstanding difficulties of the subject is the question whether any set exists with a potency between a and c. The most important theorem in this connection is that every perfect set is of potency c (p. 234). A remarkable illustration of this is that all the points within a sphere may be brought into one-one correspondence with the points on a definite straight line. As might be expected, the correspondence is not continuous.

One of the most troublesome questions discussed in this volume is that of Cantor's potency "aleph-one" (p. 135). Here we have a set the elements of which are sets of points: the element-sets recur over and over again, *qua* sets, and are treated as being all distinct in virtue of a criterion of order connected with a process of "derivation" and "deduction." Bernstein hopes to show that $\aleph_1 = c$.

The potency of an infinite set is analogous to the number of elements in a finite set. Cantor introduces symbols for potencies, calls them transfinite cardinal numbers, and investigates a calculus for them. As pointed out on p. 150, there is still a difficulty in the way of establishing comparisons between potencies really corresponding to comparisons of size between finite numbers; and, in any case, the calculus of transfinite cardinals must greatly differ from ordinary arithmetic.

Besides these ideas of order and potency, which are discussed in chapters i.-v., vi, xi, there is the additional one of content, which dominates most of the rest of the book. It must suffice to give a few illustrations to show the general nature of these inquiries. If on a line of finite length l we construct an infinite set of non-overlapping segments, their content, that is, the sum of their lengths, cannot exceed l. If it is less than l, there may be left over a set of complementary segments, but this is not necessarily the case, as is shown by the example on p. 78. Here we have a set of segments whose content is less than 0.2 of the original line, yet no segment exists on the line which does not lie wholly, or in part, within one of the selected set. The points not within any of

© 1906 Nature Publishing Group

the selected intervals form a set of potency *c*. Similar theorems apply to areas and volumes: thus, if we take a unit square it is possible to "black out" a definite region within it, the area of which may be less than any *given* quantity, in such a way that no circle, however small, can be placed within the square without covering some at least of the black region, and so that the points not within the black region form a set of potency *c*. It should be specially noticed that although the black region is, or may most conveniently be, constructed by an infinite process, it is perfectly definite, in the sense that we can say whether any given point (x, y) lies within it or not.

Another point to which attention may be called is the definition of "curve" (p. 219) as a set of points having certain properties. This is quite distinct from the idea of a path traced by a continuously moving point, and leads to very curious and interesting problems relating to rectification and quadrature.

Enough has been said to show how interesting and novel are the contents of this treatise; the examples and figures are of great help in making the general arguments intelligible, and the bibliography will enable the reader to consult all the literature relating to the subject. Finally, the appendix should not be overlooked, as it contains some important additional matter, besides a few corrections. To the latter may be added "chap iv., § 20," instead of "chap. iv., § 3," on p. 77. G. B. M.

METAMORPHOSES OF PLANTS.

Jugendform und Blütenreife im Pflanzenreich. By Dr. L. Diels. Pp. 130; with 30 illustrations in text, each with several figures. (Berlin: Gebrüder Borntraeger, 1906.) Price 3.80 marks.

IN this work the author has presented much interesting information from the point of view of an inquiry into the relations between sexual maturity and the conditions attained by the vegetative organs, especially where these from any cause show tendencies to marked changes of aspect during the development of the plants. Many of his examples are drawn from his personal investigations, the inquiry having suggested itself to him during a residence in Western Australia, where peculiarities in these relations appear to be remarkably frequent, and to be often traceable to the environment. He has also made a careful study of such published works as bear on this subject, and has used the materials derived from them with good effect as regards both the facts and the causes of abnormal conditions, and the inferences to be drawn from them. His statement of the whole subject, and of the conclusions that he believes can be fairly based on what is yet known of it, is well and clearly put, and shows the need of further inquiry, as well as the risk of pressing inferences beyond their fair limits at times.

The book is deserving of attentive perusal, and will suggest to the careful reader parallels among our own plants to some of the cases described, and questions in need of investigation from the new standpoint. The book opens with a discussion of the conditions that favour the attainment of sexual maturity in the normal course of development, and also prematurely, as measured by the stage of development of the vegetative organs. By selected examples it is shown how variable are the stages of progress, when judged by the leaves especially, at which sexual reproduction may occur. Attention is also directed to the assemblages of different species that exist on dry, poor soil, and that are distinguished by the production of precocious flowers and fruits on ill-nourished dwarfs, as well as to the production of flowers in cultivated seedlings of various trees, as in the *Swietenia Mahagoni*, recorded by Mr. Hemsley (Hooker's "Icon.," 2786), where deficiency of water appears to be the exciting cause.

The unusual precocity of reproduction is most evident where there is naturally a succession of leaf-forms before the normal period of flowering is reached, some of these being regarded as characteristic of the immature, and others of the adult plant. For the definite grades in such a succession Dr. Diels employs the term "Helikomorphie," the whole being included in the "Heteroblastie" of varied forms of parts having the same morphological value. The significance of the helicomorphies, where not of the grade usually associated with flowering, is shown to differ in different plants. In some even those characteristic of the normal immature plant may not be fully exhibited. In others these may be fully shown; but those of the adult may remain undeveloped or arrested. The causes that lead to such departures from the usual course can be shown to lie in the environment in some cases; but in others they cannot yet be explained. Among the more frequent causes of arrest and retention of immature characteristics are influences that interfere with growth, such as deficiency of water in dry soils, inclement weather at high elevations, shading and overcrowding, the plants grown under such conditions showing an unusual tendency to flower while having leaf-organs of the normal immature grade.

A comparison of allied species shows that the normal youthful helicomorphy of one may be very similar to the normal adult condition of another that lives in less favourable surroundings, *e.g.* in *Ranunculus sceleratus* and *R. pygmaeus*.

The importance of a thorough comprehension of these variations, of their significance in the life of each species, and of their value as indications of relationships between species or larger groups in classification, is self-evident. Embryology has not afforded like help in systematic botany to that obtained from it in zoology. In some cases seedling plants for a brief time show characters very unlike the adults, and suggestive of the structure of less modified allied forms, as in the well-known examples of Ulex and Acacia. All such deserve careful study; and Dr. Diels has shown how they throw light on the value of "species," as in the forms of Limosella, and of "genera," as in the relation of Regnellidium to Marsilia. His discussion of the relations of certain species of Hakea, of Grevillea, and of numerous others, largely on the basis of his personal observa-

© 1906 Nature Publishing Group